CHAPTER 3

Separation axioms

BY

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Module-5: Tietze Extension Theorem

Let X be the closed unit interval and $A = (0,1)$. If we consider the function by $f(x) = \frac{1}{x}$ defined on $(0,1)$, then f cannot be extended to the closed interval $[0,1]$. In fact any continuous function defined on $[0,1]$ must be bounded. Tietze extension theorem is one of the immediate consequence of the Urysohn lemma which deals with the problem of extending a continuous real-valued function that is defined on a subspace of a space X to a continuous function defined on all of X. This theorem is important in many of the applications of topology.

Theorem 1. *Let X be a normal space; let C be a closed subspace of X. Any continuous map of C into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.*

Proof. The idea of the proof is approximation. We shall construct a sequence of continuous functions defined on the entire space X, such that the sequence converges uniformly, and such that the restriction of each function to C approximates f. Then the limit function will be continuous, and its restriction to C will equal f.

Let X be a normal space, C a closed subset, and $f : C \to \mathbb{R}$ a continuous map. Let us first consider the bounded case, i.e. say $|f(x)| \leq M$ for all $x \in C$.

Let $A_1 = \{x \in C : f(x) \geq \frac{M}{3}\}$ and $B_1 = \{x \in C : f(x) \geq -\frac{M}{3}\}$. Then A_1 and B_1 are obviously disjoint, and they are both closed subsets of C. But C is closed in X, and therefore A_1 and B_1 must be closed in X. By Urysohn's lemma we can find a continuous map $g_1 : X \to [-\frac{M}{3}, \frac{M}{3}]$ which takes the value $\frac{M}{3}$ on A_1 and $-\frac{M}{3}$ on B_1 and which takes values in $(-\frac{M}{3}, \frac{M}{3})$ on $X - (A_1 \cup B_1)$. Notice that $|f(x) - g_1(x)| \leq \frac{2M}{3}$ on C.

Now consider the function $f - g_1$ and let A_2 consist of those points of C for which $|f(x) - g_1(x)| \geq \frac{2M}{9}$, and B_2 those points for which $|f(x) - g_1(x)| \leq -\frac{2M}{9}$. Again applying

Uryshon's lemma a second time we can find a map $g_2 : X \to [-\frac{2M}{9}, \frac{2M}{9}]$ which takes the value $\frac{2M}{9}$ on A_2 and $-\frac{2M}{9}$ on B_2, and values in $(-\frac{2M}{9}, \frac{2M}{9})$ on the remaining points of X. If we compute $f(x) - g_1(x) - g_2(x)$, we see that $|f(x) - g_1(x) - g_2(x)| \leq \frac{4M}{9}$ on C.

By repeating this process we can construct a sequence of maps $g_n : X \to [-\frac{2^{n-1}M}{3^n}, \frac{2^{n-1}M}{9}]$ which satisfy:

(a) $|f(x) - g_1(x) - g_2(x) - \ldots g_n(x)| \leq \frac{2^n M}{3^n}$ on C; and
(b) $|g_n(x)| < \frac{2^{n-1}M}{3^n}$ on $X - C$.

We now define

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

for all x in X. The convergency of the infinite series follows from the comparison theorem of calculus, it converges by comparison with the geometric series

$$\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}.$$

To show that g is continuous, we must show that the sequence s_n of partial sums converges to g uniformly. This fact follows at once from the "Weierstrass M-test".

Finally we show that $g(x) = f(x)$ for $x \in C$. $g(x)$ is by definition the limit of the infinite sequence $s_n(x)$ of partial sums. Since

$$|f(x) - \sum_{i=1}^{n} g_i(x)| = |f(x) - s_n(x)| \leq \left(\frac{2}{3}\right)^n$$

for all x in C, it follows that $s_n(x) \to f(x)$ for all $x \in C$. Therefore, f and g agree on C.

If $|g(x)|$ is bounded then $|g(x)| = \sum_{n=1}^{\infty} |g_n(x)| \leq \sum_{n=1}^{\infty} M\frac{2^{n-1}}{3^n} = M$, and $|g(x)|$ is strictly less than M on $X - C$ by (b).

Let $f : C \to \mathbb{R}$ be arbitrary continuous function. Let us choose a homeomorphism h from the real line to the interval $(-1, 1)$ and consider the composition $h \circ f$, which is bounded. Therefore by the above argument we can extend it to a continuous real-valued function g on X, all of whose values lie strictly between -1 and 1. So the composition $h^{-1} \circ g$ is well defined, and by construction it extends f over X. This completes the proof. □

The following corollary can be obtained easily.

Corollary 1. *Let X be a normal space; let C be a closed subspace of X. Then Any continuous map of C into the closed interval $[a, b]$ may be extended to a continuous map of all of X into $[a, b]$.*

Proof. Proof follows from the proof of main Theorem. □

Though we have used Urysohn Lemma in the proof Tietz extension Theorem, but if a space satisfies Tietz extension Theorem, then it satisfies Urysohn Lemma.

Proposition 1. *Show that the Tietze extension theorem implies the Urysohn lemma.*

Proof. We need to use pasting lemma and then to apply Tietze extension theorem. □

Definition 1. *A space Y is said to have the universal extension property if for any given normal space X, any closed subset A of X, and any continuous function $f : A \to Y$, there exists an extension of f to a continuous map of X into Y.*

Proposition 2. *Prove that $\mathbb{R}^J$ has the universal extension property.*

Proof. Consider a normal space X, a closed subset A of X, and a continuous function $f : A \to Y$. The for each $i \in J$, $f_i = \pi_i \circ f : A \to \mathbb{R}$ is continuous and hence has a continuous extension say g_i over X. Then $g = (g_i)_i \in \mathbb{R}^J$ is a continuous extension of f over X. □

Definition 2. *Let X be a topological space. If Y is a subspace of Z, we say that Y is a retract of X if there is a continuous map $r : X \to Y$ such that $r(y) = y$ for each $y \in Y$.*

It is easy to observe that if X is Hausdorff then any retract Y of X is closed.

Example 1. *$\mathbb{S}^1$ is a retract of $\mathbb{R}^2 - \{\bar{0}\}$. The map $r : \mathbb{R}^2 - \{\bar{0}\} \to \mathbb{S}^1$ defined by $r(x) = \frac{x}{\|x\|}$ is a retraction.*

Example 2. *If Y is homeomorphic to a retract of R^J, then Y has the universal extension property.*

Definition 3. *Let X be a normal space. Then X is said to be an absolute retract if for any normal space Y and any closed subspace Y_0 of Y, homeomorphic to X, the space Y_0 is a retract of Y.*

Proposition 3. *If X has the universal extension property, then X is an absolute retract.*

Proof. Let Y be a normal space, Y_0 be a closed subspace of Y homeomorphic to X. Let $f : Y_0 \to X$ be the homeomorphism. Since X has the universal extension property there exists a continuous extension say $g : Y \to X$ of $f : Y_0 \to X$. Then $r = f^{-1} \circ g$ is the required retraction. $\square$